The mechanics and control for multi-particle systems

1998 J. Phys. A: Math. Gen. 31 3849

(http://iopscience.iop.org/0305-4470/31/16/013)

J. Phys. A: Math. Gen. **31** (1998) 3849–3865. Printed in the UK PII: S0305-4470(98)83213-7

The mechanics and control for multi-particle systems

Toshihiro Iwai
Department of Applied Mathematics and Physics, Kyoto University, Kyoto 606-01, Japan

Received 8 April 1997, in final form 5 January 1998

Abstract. This paper deals with the mechanics and control for multi-particle systems from a geometric point of view. The centre-of-mass system is viewed as a principal fibre bundle with structure group $SO(3)$, the base space of which is called the internal or shape space. A natural connection and a natural Riemannian metric are both defined on the centre-of-mass system. The equations of motion for the multi-particle system are derived in the Lagrangian formalism adapted to the bundle structure, and then reduced with the conserved total angular momentum. In contrast with this, the control problem is studied with non-holonomic constraints, i.e. with the vanishing total angular momentum, and equations of motion are determined for an optimally controlled multi-particle system. The resultant equations derived in each of the mechanical and control systems are to be compared.

1. Introduction

A geometric way to the mechanics for multi-particle systems is to treat the centre-of-mass system as a principal fibre bundle. It was Guichardet (1984) who first defined rotational and vibrational vectors strictly, and thereby showed that rotations cannot be separated from vibrations on the basis of the connection theory applied for the centre-of-mass system as a principal fibre bundle. The author (1987a) set up Hamiltonian formalism for multi-particle systems, in a rather abstract way, also on the basis of the connection theory. In this paper, however, the equations of motion are derived in Lagrangian formalism in terms of local coordinates, and reduced along with the conserved total angular momentum. As for multi-particle systems, Montgomery (1990, 1991) treated the falling Cat problem, a control problem with non-holonomic constraints, from the viewpoint of a bundle picture, i.e. from the gauge theoretical point of view. Recently, Littlejohn and Reinsch (1997) studied multi-particle systems also from the bundle picture viewpoint. This paper also deals with the falling Cat problem in order to compare its equations of motion with those for the reduced equations of motion for the mechanical system.

The paper is organized as follows: section 2 contains the setting up of the centre-of-mass system as a principal $SO(3)$ bundle, on which a natural connection and a natural metric are defined, and thereby rotational and vibrational vectors are defined strictly. Miscellaneous related formulae will also be given. In section 3, the equations of motion for the multi-particle system are derived in the Lagrangian formalism, and then reduced by the use of the conserved total angular momentum. The reduced equations consist of two sets; one is mainly concerned with angular variables, and the other with internal coordinates. Section 4 deals with an optimal control problem for the multi-particle system, in which the multi-particle system is operated so that the vibrational energy of the system is minimized with the constraint of the vanishing total angular momentum. The equations of motion to which the optimally controlled multi-particle system is subject is determined on the maximum principle. The resultant equations are compared with those obtained in section 3. Section 5 contains remarks on the Lagrangian and Hamiltonian formalisms adapted to the bundle structure.

2. Geometry of the centre-of-mass system

Let X_0 be the space of all the ennuples $x = (x_1, \dots, x_N)$ of particle position vectors $x_\alpha \in \boldsymbol{R}^3$, each particle having mass $m_\alpha, \alpha = 1, \dots, N$. As is well known, the translational degrees of freedom are removed from X_0 to give rise to the centre-of-mass system

$$X = \left\{ x = (x_1, \dots, x_N) \,\middle|\, \sum_{\alpha=1}^{N} m_\alpha x_\alpha = 0 \right\}. \tag{2.1}$$

The rotation group $SO(3)$ acts on X in a natural manner

$$\Phi_g : x \mapsto gx := (gx_1, \dots, gx_N) \qquad g \in SO(3),\ x \in X. \tag{2.2}$$

We assume here that the configurations of N particles are not rectilinear, i.e. we restrict X to the subset at each point of which

$$F_x = \mathrm{span}\{x_1, x_2, \dots, x_N\} \tag{2.3}$$

is of dimension greater than or equal to two; $\dim F_x \geqslant 2$. Let

$$\dot{X} = \{x \in X \,|\, \dim F_x \geqslant 2\} \tag{2.4}$$

then the compact group $SO(3)$ acts freely on $\dot{X}$, so that the quotient space $\dot{X}/SO(3)$ becomes a manifold (see Abraham and Marsden (1978)). Thus $\dot{X}$ is made into a principal fibre bundle (see Cushman and Bates (1997), for example)

$$\pi : \dot{X} \longrightarrow M := \dot{X}/SO(3). \tag{2.5}$$

The base space M is referred to as the internal space or shape space, the dimension of which is, of course, $n := 3N - 6$, since $\dim \dot{X} = 3N - 3$. Let U be an open subset of M. Then the local triviality, $\pi^{-1}(U) \cong U \times SO(3)$, of the $SO(3)$ bundle (2.5) is expressed as

$$x = g\sigma(q) \qquad \sigma(q) = (\sigma_\alpha(q)) = \left(\sum_{a=1}^{3} C_\alpha^a(q) e_a \right) \qquad (q, g) \in U \times SO(3) \tag{2.6}$$

where $\sigma : U \to \dot{X}$ is a local section, and $e_a, a = 1, 2, 3$, are the standard basis of $\boldsymbol{R}^3$. We notice here that $\sigma(q)$ denotes a way to put the multi-particle system with the shape determined by $q \in U$, in the space $\boldsymbol{R}^3$. Let V be another open subset of M with $U \cap V \neq \emptyset$. Then one has another local section $\tau : V \to \dot{X}$ such that

$$x = h\tau(q) \qquad (q, h) \in V \times SO(3). \tag{2.7}$$

The local sections σ and τ are then related, on $U \cap V$, by

$$\tau = k\sigma \qquad k = h^{-1}g \tag{2.8}$$

where $k = k(q)$ and $q \in U \cap V$. Note also that $\dot{X}$ becomes a trivial bundle for three-particle systems (Iwai 1987b), so that $\sigma : U \to \dot{X}$ can be defined globally for those systems.

The centre-of-mass system $\dot{X}$ is endowed with a metric $\mathrm{d}s^2$, which is defined, at $x = (x_1, \dots, x_N) \in X$, to be

$$\mathrm{d}s^2 = \sum_{\alpha=1}^{N} m_\alpha (\mathrm{d}x_\alpha | \mathrm{d}x_\alpha) \tag{2.9}$$

where $(\,|\,)$ denotes the standard inner product in $\boldsymbol{R}^3$. To define a natural connection on $\dot{X}$, we start by setting up some notations necessary for the definition. Let $R : \boldsymbol{R}^3 \to so(3)$ and $A_x : \boldsymbol{R}^3 \to \boldsymbol{R}^3$ be the vector space isomorphism and the inertia tensor, each of which are defined to be

$$R(w) = \begin{pmatrix} 0 & -w^3 & w^2 \\ w^3 & 0 & -w^1 \\ -w^2 & w^1 & 0 \end{pmatrix} \qquad \text{for } w = (w^a) \in \boldsymbol{R}^3 \tag{2.10}$$

and

$$A_x(w) = \sum_{\alpha=1}^{N} m_\alpha x_\alpha \times (w \times x_\alpha) \qquad x \in \dot{X},\ w \in \boldsymbol{R}^3 \tag{2.11}$$

respectively. We notice here that A_x is a symmetric positive-definite operator for x with $\dim F_x \geqslant 2$, so that A_x^{-1} exists. For further calculations we carry out later, let us be reminded of the fundamental properties of R and A_x

$$R(w)z = w \times z \qquad w, z \in \boldsymbol{R}^3 \tag{2.12}$$

$$R(gw) = gR(w)g^{-1} \qquad w \in \boldsymbol{R}^3,\ g \in SO(3) \tag{2.13}$$

$$R(v) \cdot R(w) = (v|w) \qquad v, w \in \boldsymbol{R}^3 \tag{2.14}$$

$$A_{gx}(w) = gA_x(g^{-1}w) =: \mathrm{Ad}_g A_x(w) \qquad w \in \boldsymbol{R}^3,\ g \in SO(3). \tag{2.15}$$

Here equation (2.14) defines the inner product in $so(3)$. The connection form ω (see Iwai (1987a, b)) is then defined to be

$$\omega = R\left(A_x^{-1} \sum_{\alpha=1}^{N} m_\alpha x_\alpha \times \mathrm{d}x_\alpha \right). \tag{2.16}$$

On this set-up, the rotational vectors are defined to be infinitesimal generators of the $SO(3)$ action, which take the form

$$\frac{\mathrm{d}}{\mathrm{d}t} \exp(tR(w))x|_{t=0} = (w \times x_1, \ldots, w \times x_N) \tag{2.17a}$$

or in terms of differential operators

$$\sum_{\alpha=1}^{N} \left(w \times x_\alpha \,\middle|\, \frac{\partial}{\partial x_\alpha} \right) = \left(w \,\middle|\, \sum_{\alpha=1}^{N} x_\alpha \times \frac{\partial}{\partial x_\alpha} \right) = (w|J) \qquad J = \sum_{\alpha=1}^{N} x_\alpha \times \frac{\partial}{\partial x_\alpha} \tag{2.17b}$$

where J is the total angular momentum operator. A tangent vector $v = (v_1, \ldots, v_N)$ to X at x is called a vibrational vector, if it is orthogonal to any rotational vector at x with respect to the metric (2.9). Hence, v is a vibrational (or horizontal) vector, if and only if

$$\sum_{\alpha=1}^{N} m_\alpha x_\alpha \times v_\alpha = 0 \tag{2.18}$$

which is equivalent to $\omega(v) = 0$. Moreover, we have to point out that ω satisfies, for rotational vectors

$$\omega\left(\sum_{\alpha=1}^{N} \left(w \times x_\alpha \,\middle|\, \frac{\partial}{\partial x_\alpha} \right) \right) = R(w) \qquad w \in \boldsymbol{R}^3 \tag{2.19}$$

and is subject to the transformation

$$\omega_{gx} = \mathrm{Ad}_g \omega_x \qquad g \in SO(3) \tag{2.20}$$

the proof of which can be carried out in a straightforward manner.

We proceed now to describe the rotational and vibrational vectors in local coordinates with respect to the local triviality (2.6). Let ω^a and J_a be the components of ω and of J with respect to the fixed frame e_a, respectively

$$\omega = \sum_{a=1}^{3} R(e_a)\omega^a \qquad \omega \cdot R(e_a) = \omega^a \tag{2.21}$$

$$J = \sum_{a=1}^{3} e_a J_a \qquad J_a = (e_a|J). \tag{2.22}$$

The forms ω^a and $\mathrm{d}q^i$ constitute a local basis of the space of one-forms on $\dot{X}$, $a = 1, 2, 3$, $i = 1, 2, \ldots, n = \dim M$. To be strict in notation, we have to use $\pi^*\mathrm{d}q^i$, the pull-back of $\mathrm{d}q^i$, for $\mathrm{d}q^i$, but we use $\mathrm{d}q^i$ for notational simplicity. Furthermore, the vector fields J_a and $\partial_i^* := (\partial/\partial q^i)^*$, the horizontal lift of $\partial/\partial q^i$ defined by $\omega(\partial_i^*) = 0$ and $\pi_*\partial_i^* = \partial/\partial q^i$, are a local basis of the space of vector fields, where π_* is the differential of π. Then one has

$$\begin{aligned} &\omega^a(J_b) = \delta_b^a \qquad \mathrm{d}q^i(J_b) = 0 \\ &\omega^a(\partial_j^*) = 0 \qquad \mathrm{d}q^i(\partial_j^*) = \delta_j^i. \end{aligned} \tag{2.23}$$

For the local expression of ω^a, we write out (2.16) in the local coordinates given in (2.6). A calculation then results in

$$\omega^a = \Theta^a + \sum_{i=1}^{n} \beta_i^a \,\mathrm{d}q^i \tag{2.24}$$

where we have set

$$\mathrm{d}g g^{-1} = \sum_{a=1}^{3} \Theta^a R(e_a) \tag{2.25}$$

$$\beta_i^a = \left(A_x^{-1} \sum_{\alpha=1}^{N} m_\alpha x_\alpha \times \frac{\partial x_\alpha}{\partial q^i} \,\middle|\, e_a \right). \tag{2.26}$$

The Θ^a and J_a are expressed in terms of Euler angles, which we need not give explicitly here. As for the local expression of $\partial_i^* = (\partial/\partial q^i)^*$, we obtain, from (2.23)

$$\partial_i^* = \frac{\partial}{\partial q^i} - \sum_{a=1}^{3} \beta_i^a J_a. \tag{2.27}$$

Now we have to note that the transformation (2.20) to which ω is subject implies that β_i^a are subject to the transformation

$$\beta_i^a(gx) = \sum_{b=1}^{3} g_{ab}\beta_i^b(x) \qquad g = (g_{ab}) \in SO(3). \tag{2.28}$$

The infinitesimal version of (2.28) with $g = \exp(tR(e_c))$ is expressed as

$$J_c(\beta_i^a) = -\sum_{b=1}^{3} \varepsilon_{cab}\beta_i^b \tag{2.29}$$

where ε_{cab} is the antisymmetric symbol with $\varepsilon_{123} = 1$. Moreover, from (2.15) the components of the inertia tensor, $A_{ab}(x) := (e_a|A_x(e_b))$, are shown to be subject to the transformation

$$A_{ab}(gx) = \sum_{c,d} g_{ac}A_{cd}(x)g_{bd} \qquad g = (g_{ab}) \in SO(3). \tag{2.30}$$

The infinitesimal transformation of (2.30) for $g = \exp tR(e_c)$ proves to be given by

$$J_c(A_{ab}) = [R(e_c), A]_{ab} = \sum_d \varepsilon_{cda} A_{db} + \sum_d \varepsilon_{cdb} A_{ad} \tag{2.31}$$

where $A = (A_{ab})$.

We now wish to express the metric $\mathrm{d}s^2$ in terms of ω^a and $\mathrm{d}q^i$. From (2.17) it follows that $J_a x_\alpha = e_a \times x_\alpha$, so that from (2.9) and (2.11), one obtains

$$\mathrm{d}s^2(J_a, J_b) = A_{ab}(x). \tag{2.32}$$

Further, since rotational vector fields J_a and vibrational vector fields ∂_i^* are orthogonal, the metric $\mathrm{d}s^2$ turns out to be expressed as

$$\mathrm{d}s^2 = \sum_{a,b=1}^{3} A_{ab}\omega^a\omega^b + \sum_{i,j=1}^{n} a_{ij}\, \mathrm{d}q^i\, \mathrm{d}q^j \tag{2.33}$$

where

$$a_{ij} := \mathrm{d}s^2(\partial_i^*, \partial_j^*). \tag{2.34}$$

It is to be noted that since $\mathrm{d}s^2$ is invariant under the $SO(3)$ action, and since the vibrational vector field ∂_j^* is in one-to-one correspondence with $\partial_j = \partial/\partial q^j$, a tangent vector field on $U \subset M$, equation (2.34) defines a metric tensor a_{ij} on the internal space M.

We finally proceed to the curvature form, which is defined to be

$$\Omega = \sum_{a=1}^{3} R(e_a)\Omega^a := \mathrm{d}\omega - \omega \wedge \omega. \tag{2.35}$$

Then a calculation provides

$$\Omega^c = \mathrm{d}\omega^c - \sum_{a<b} \varepsilon_{abc}\omega^a \wedge \omega^b = \sum_{i<j} F_{ij}^c\, \mathrm{d}q^i \wedge \mathrm{d}q^j \tag{2.36}$$

where

$$F_{ij}^c = \frac{\partial \beta_j^c}{\partial q^i} - \frac{\partial \beta_i^c}{\partial q^j} - \sum_{a,b=1}^{3} \varepsilon_{abc}\beta_i^a\beta_j^b. \tag{2.37}$$

In addition, we obtain the transformation property of the curvature form. From (2.20) and (2.35) it follows that

$$\Omega_{gx} = \mathrm{Ad}_g \Omega_x. \tag{2.38}$$

Then the components $\Omega^a = (F_{ij}^a)$ are subject to the transformation

$$F_{ij}^c(gx) = \sum_{a=1}^{3} g_{ca} F_{ij}^a(x) \qquad g = (g_{ab}) \in SO(3). \tag{2.39}$$

The curvature tensor F_{ij}^a is also introduced in terms of vector fields; on using (2.22), (2.27) and (2.29), the rotational vector fields J_a and the vibrational vector fields ∂_i^* are shown to satisfy the following commutation relations

$$[J_a, J_b] = -\sum_{c=1}^{3} \varepsilon_{abc} J_c \qquad [\partial_i^*, \partial_j^*] = -\sum_{c=1}^{3} F_{ij}^c J_c \qquad [\partial_i^*, J_a] = 0. \tag{2.40}$$

The middle equation of (2.40) means that the two independent vibrational vectors, ∂_i^* and ∂_j^*, are coupled to give rise to an infinitesimal rotation. This fact implies that vibrations cannot be separated from rotations. Another implication is that the distribution spanned by

$\{\partial_i^*\}$ is not completely integrable in the sense of Frobenius (see Matsushima (1972)), so that there are no submanifolds to which ∂_i^* are tangent. If there were such a submanifold, only vibrational motions would take place on it, and it would be able to be identified with (an open submanifold of) the internal space M. In terms of mechanics, the constraint of the vanishing total angular momentum (see (2.18)) is equivalent to assigning the distribution spanned by $\{\partial_i^*\}$, and these facts mean that this constraint is non-holonomic.

3. Equations of motion

In this section, we aim to obtain the equations of motion for multi-particle systems in the Lagrangian formalism adapted for the bundle structure of the centre-of-mass system X. To this end, the Lagrangian formalism in terms of 'quasi-coordinates' (see Whittaker (1937)) is of great use. We start with a brief review of the Lagrangian formalism adapted for our purpose (see also Naimark and Fufaev (1972), and Koiller (1992)). For local expressions of the equations of motion, it is sufficient for us to work in an open subset W of $\boldsymbol{R}^{3N-3}$. Let ξ^λ, $\lambda = 1, 2, \ldots, 3N-3$, be a local coordinate system in W. Let X_λ and θ^λ be a local basis of vector fields and its dual on W, respectively, which are denoted by

$$X_\lambda = \sum_\mu B_\lambda^\mu \frac{\partial}{\partial \xi^\mu} \qquad \theta^\lambda = \sum_\mu A_\mu^\lambda \, \mathrm{d}\xi^\mu \tag{3.1}$$

respectively, with $\sum_\lambda A_\lambda^\mu B_\nu^\lambda = \delta_\nu^\mu$. Then, one has after differentiation

$$\mathrm{d}\theta^\lambda = \sum_{\sigma<\kappa} \gamma_{\sigma\kappa}^\lambda \theta^\kappa \wedge \theta^\sigma \qquad \gamma_{\sigma\kappa}^\lambda := \sum_{\mu,\nu} \left(\frac{\partial A_\mu^\lambda}{\partial \xi^\nu} - \frac{\partial A_\nu^\lambda}{\partial \xi^\mu} \right) B_\sigma^\mu B_\kappa^\nu. \tag{3.2}$$

It is clear that $\gamma_{\sigma\kappa}^\lambda$ is anti-symmetric in σ and κ.

Let

$$\dot{\pi}^\lambda = \sum_\mu A_\mu^\lambda(\xi) \dot{\xi}^\mu. \tag{3.3}$$

The equations of motion can be described in the Lagrangian formalism, in terms of $\dot{\pi}^\lambda$ and ξ^λ. We express the Lagrangian $L(\xi, \dot{\xi})$ as

$$L^*(\xi, \dot{\pi}) = L(\xi, \dot{\xi}). \tag{3.4}$$

Then the usual Lagrangian equations of motion in terms of $(\xi, \dot{\xi})$

$$\frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\partial L}{\partial \dot{\xi}^\lambda} \right) - \frac{\partial L}{\partial \xi^\lambda} = 0 \qquad \lambda = 1, \ldots, 3N-3 \tag{3.5}$$

are put in the form

$$\frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\partial L^*}{\partial \dot{\pi}^\sigma} \right) - X_\sigma L^* + \sum_{\mu,\kappa} \gamma_{\sigma\kappa}^\mu \frac{\partial L^*}{\partial \dot{\pi}^\mu} \dot{\pi}^\kappa = 0 \qquad \sigma = 1, \ldots, 3N-3 \tag{3.6}$$

where X_σ and $\gamma_{\sigma\kappa}^\mu$ are the vector fields and the coefficients given in (3.1) and (3.2), respectively.

We now apply these equations to our multi-particle system in the open subset $\pi^{-1}(U) \cong U \times SO(3)$ referred to in (2.6). From (2.23), the system of one-forms is given by

$$\theta^a = \omega^a \qquad \theta^{3+i} = \mathrm{d}q^i \qquad a = 1, 2, 3, \; i = 1, \ldots, n = \dim M \tag{3.7}$$

and the dual system of vector fields is written as

$$X_a = J_a \qquad X_{3+i} = \partial_i^* \qquad a = 1, 2, 3, \; i = 1, \ldots, n = \dim M. \tag{3.8}$$

Then, equation (2.36) and $\mathrm{d}(\mathrm{d}q^i) = 0$ provide, when compared with (3.2)

$$\gamma^a_{bc} = -\varepsilon_{bca} \qquad \gamma^a_{3+i,3+j} = -F^a_{ij} \tag{3.9}$$

with the other $\gamma^\lambda_{\mu\nu}$ all vanishing. To express the Lagrangian L^*, we introduce variables $\dot{\pi}^\lambda$ according to (3.3) by

$$\dot{\pi}^a = \omega^a_t \qquad \dot{\pi}^{3+i} = \dot{q}^i \tag{3.10}$$

where ω^a_t are defined through (2.24) as

$$\omega^a_t := \omega^a\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) = \Theta^a_t + \sum_i \beta^a_i \dot{q}^i \qquad \Theta^a_t = \Theta^a\left(\frac{\mathrm{d}}{\mathrm{d}t}\right). \tag{3.11}$$

Then, from (2.33) together with a potential function V, one has the Lagrangian

$$L^* = \frac{1}{2}\sum_{a,b} A_{ab}\omega^a_t\omega^b_t + \frac{1}{2}\sum_{i,j} a_{ij}\dot{q}^i\dot{q}^j - V. \tag{3.12}$$

The application of (3.6) to (3.12) along with (3.9) provides

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L^*}{\partial \omega^a_t}\right) - J_a L^* - \sum_{b,c}\varepsilon_{acb}\frac{\partial L^*}{\partial \omega^b_t}\omega^c_t = 0 \tag{3.13}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L^*}{\partial \dot{q}^i}\right) - \partial^*_i L^* - \sum_a\sum_j F^a_{ij}\frac{\partial L^*}{\partial \omega^a_t}\dot{q}^j = 0. \tag{3.14}$$

Equation (3.13) turns out to be expressed, in vector notation with $A = (A_{ab})$ and $\omega_t = (\omega^a_t)$, as

$$\frac{\mathrm{d}}{\mathrm{d}t}(A\omega_t) - A\omega_t \times \omega_t + JV - \omega_t \times A\omega_t = 0 \tag{3.15}$$

where $JV = \sum_a e_a J_a V$, and we have used the formula (2.31). Since the total angular momentum is expressed as $L = \sum_\alpha m_\alpha x_\alpha \times \dot{x}_\alpha = A\omega_t$, as is easily seen from (2.16), equation (3.15) is put in the form

$$\frac{\mathrm{d}}{\mathrm{d}t}L = -JV. \tag{3.16}$$

If the potential is rotational invariant, this equation implies conservation of the total angular momentum. On the other hand, equation (3.14) becomes expressed as

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\sum_j a_{ij}\dot{q}^j\right) - \frac{1}{2}\sum_{k,j}\frac{\partial a_{kj}}{\partial q^i}\dot{q}^k\dot{q}^j - \frac{1}{2}\sum_{a,b}\partial^*_i A_{ab}\omega^a_t\omega^b_t - \sum_j\sum_{a,b}F^a_{ij}A_{ab}\omega^b_t\dot{q}^j + \frac{\partial V}{\partial q^i} = 0 \tag{3.17}$$

which proves to be equivalent to

$$\frac{\mathrm{d}^2 q^i}{\mathrm{d}t^2} + \sum_{j,k}\begin{Bmatrix} i \\ j\ k\end{Bmatrix}\frac{\mathrm{d}q^j}{\mathrm{d}t}\frac{\mathrm{d}q^k}{\mathrm{d}t} = \frac{1}{2}\sum_k\sum_{a,b}\partial^*_k A_{ab}a^{ik}\omega^a_t\omega^b_t - \sum_{j,k}\sum_c A_{ab}F^b_{jk}a^{ik}\frac{\mathrm{d}q^j}{\mathrm{d}t}\omega^a_t - \sum_j a^{ij}\frac{\partial V}{\partial q^j} \tag{3.18}$$

where $(a^{ij}) = (a_{ij})^{-1}$. Thus we have obtained two systems of equations, (3.16) and (3.18), which are the equations of motion for the multi-particle system.

In what follows, we assume that the potential V is rotational invariant, so that the total angular momentum is conserved. Hence we treat L as a constant vector, and thereby reduce the equations of motion (3.18). Equation (3.18) along with $L = A\omega_t$ can then be put in the form

$$\frac{\mathrm{d}^2 q^i}{\mathrm{d}t^2} + \sum_{j,k} \left\{ \begin{matrix} i \\ j\,k \end{matrix} \right\} \frac{\mathrm{d}q^j}{\mathrm{d}t} \frac{\mathrm{d}q^k}{\mathrm{d}t} = -\frac{1}{2} \sum_k \sum_{a,b} a^{ik} \partial_k^* A^{ab} L_a L_b - \sum_{j,k} \sum_c a^{ik} L_c F_{jk}^c \frac{\mathrm{d}q^j}{\mathrm{d}t} - \sum_j a^{ij} \frac{\partial V}{\partial q^j} \tag{3.19}$$

where use has been made of $\sum_b A^{ab} A_{bc} = \delta_{ac}$. This equation was also found by Littlejohn and Reinsch (1996).

What we note about (3.19) is that this equation is not in a closed form, if L is fixed during the motion. In fact, the right-hand side contains angular variables, i.e. depends on $SO(3)$ through A^{ab} and F_{ij}^c, but the left-hand side of (3.19) is independent of $SO(3)$. This implies that we need another equation for angular variables in order to obtain equations of motion in the closed form. However, we observe that, if L is constant, the right-hand side of (3.19) is invariant under the rotation about L, i.e. under the action of $h \in SO(3)$ satisfying $hL = L$. This is because $\partial_k^* A^{-1} = (\partial_k^* A^{ab})$ is subject to the transformation

$$(\partial_k^* A^{-1})_{gx} = \mathrm{Ad}_g (\partial_k^* A^{-1})_x \tag{3.20}$$

the same transformation as A, and because (F_{jk}^c) is subject to (2.39). Here equation (3.20) is a consequence of the fact that ∂_k^* is invariant under the $SO(3)$ action, $\Phi_{g*}\partial_k^* = \partial_k^*$, the infinitesimal version of which is $[\partial_k^*, J_a] = 0$, the last equation of (2.40). Hence, we need in reality equations for angular variables which do not keep L invariant. Since the set of $h \in SO(3)$ satisfying $hL = L$, $L \neq 0$, forms a subgroup $SO(2)$, the angular variables we need lie on the sphere $S^2 \simeq SO(3)/SO(2)$. To find equations on S^2, we consider the vector defined by

$$\Lambda := g^{-1} L \tag{3.21}$$

where $g \in SO(3)$ is the angular variable introduced in (2.6). The Λ is an analogue to the body-fixed angular momentum for a rigid body. The magnitude of this vector is, of course, conserved; $||\Lambda|| = ||L|| =$ constant, and hence Λ varies in the sphere S^2. A calculation along with

$$\dot{g} g^{-1} = R(\Theta_t) \qquad \Theta_t := \Theta\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) \tag{3.22}$$

shows that Λ is subject to the equation

$$\frac{\mathrm{d}\Lambda}{\mathrm{d}t} = -g^{-1}\Theta_t \times \Lambda. \tag{3.23}$$

Since $L = A_{g\sigma(q)}\omega_t$, equation (3.21) is expressed as

$$\Lambda = A_{\sigma(q)} g^{-1} \omega_t = A_{\sigma(q)} \left(g^{-1}\Theta_t + \sum_i \beta_i(\sigma(q)) \frac{\mathrm{d}q^i}{\mathrm{d}t} \right) \tag{3.24}$$

where

$$\beta_i(\sigma(q)) = \sum_a \beta_i^a(\sigma(q)) e_a. \tag{3.25}$$

Then, equation (3.23) is rewritten as

$$\frac{\mathrm{d}\Lambda}{\mathrm{d}t} = -(A_{\sigma(q)}^{-1}\Lambda) \times \Lambda + \sum_i \frac{\mathrm{d}q^i}{\mathrm{d}t} (\beta_i(\sigma(q)) \times \Lambda). \tag{3.26}$$

This is the equation for Λ, depending on internal coordinates.

We note here that Λ is a locally-defined variable, so that we need to verify that the equation for Λ is independent of the choice of local sections. According to the local triviality $\pi^{-1}(V) \cong V \times SO(3)$ referred to in (2.7), we have to take

$$\overline{\Lambda} = h^{-1}L \tag{3.27}$$

in place of Λ. Then, from (2.8), $\overline{\Lambda}$ and Λ are related by

$$\overline{\Lambda} = k\Lambda. \tag{3.28}$$

Further, we put

$$R(\overline{\Theta}_t) = \dot{h}h^{-1} \tag{3.29}$$

which corresponds to (3.22). Then, equation (2.8) implies that

$$R(\overline{\Theta}_t) = R(\Theta_t) - g(k^{-1}\dot{k})g^{-1}. \tag{3.30}$$

From (3.28)–(3.30) it follows that

$$\frac{\mathrm{d}\overline{\Lambda}}{\mathrm{d}t} = -h^{-1}\overline{\Theta}_t \times \overline{\Lambda} \tag{3.31}$$

which is the equation that $\overline{\Lambda}$ is expected to hold from (3.23). As a consequence, one also obtains

$$\frac{\mathrm{d}\overline{\Lambda}}{\mathrm{d}t} = -(A^{-1}_{\tau(q)}\overline{\Lambda}) \times \overline{\Lambda} + \sum_i \frac{\mathrm{d}q^i}{\mathrm{d}t}(\beta_i(\tau(q)) \times \overline{\Lambda}). \tag{3.32}$$

Thus, equation (3.26) turns out to be independent of the choice of local sections. We can verify (3.32) also by the use of (3.28) and the 'gauge' transformation

$$\sum_i R(\beta_i(\tau(q)))\,\mathrm{d}q^i = \mathrm{d}kk^{-1} + \sum_i R(k\beta_i(\sigma(q)))\,\mathrm{d}q^i \tag{3.33}$$

which comes from (2.24) along with the local sections, τ and σ, given in (2.8).

We return to equation (3.19). On taking the local section σ, equation (3.19) turns out to be expressed as

$$\begin{aligned}\frac{\mathrm{d}^2q^i}{\mathrm{d}t^2} + \sum_{j,k}\left\{\begin{matrix} i \\ j\;k\end{matrix}\right\}\frac{\mathrm{d}q^j}{\mathrm{d}t}\frac{\mathrm{d}q^k}{\mathrm{d}t} &= -\frac{1}{2}\sum_j\sum_{a,b} a^{ij}(\partial^*_j A^{ab})_{\sigma(q)}\Lambda_a\Lambda_b \\ &\quad - \sum_{j,k}\sum_c a^{ik}\Lambda_c F^c_{jk}(\sigma(q))\frac{\mathrm{d}q^j}{\mathrm{d}t} - \sum_j a^{ij}\frac{\partial V}{\partial q^j}.\end{aligned} \tag{3.34}$$

We have to verify that the right-hand side of (3.34) is independent of the choice of local sections as well. The quantity $(\partial^*_j A^{-1})_{\sigma(q)} = (\partial^*_j A^{ab})_{\sigma(q)}$ in the first term of the right-hand side of (3.34) turns out to be put in the form

$$(\partial^*_j A^{-1})_{\sigma(q)} = \frac{\partial A^{-1}_{\sigma(q)}}{\partial q^j} - [R(\beta_j(\sigma(q))), A^{-1}_{\sigma(q)}] \tag{3.35}$$

which can be verified by using (2.27) and

$$J_a(A^{-1}) = [R(e_a), A^{-1}] \tag{3.36}$$

a consequence of (2.31). For the local sections τ and σ, the right-hand side of (3.35) transforms according to

$$\frac{\partial A^{-1}_{\tau(q)}}{\partial q^j} - [R(\beta_j(\tau(q))), A^{-1}_{\tau(q)}] = \mathrm{Ad}_{k(q)}\left(\frac{\partial A^{-1}_{\sigma(q)}}{\partial q^j} - [R(\beta_j(\sigma(q))), A^{-1}_{\sigma(q)}]\right) \tag{3.37}$$

which can be proved by the use of $A_\tau^{-1} = kA_\sigma^{-1}k^{-1}$ and (3.33). Thus one has

$$(\partial_j^* A^{-1})_{\tau(q)} = \mathrm{Ad}_{k(q)}(\partial_j^* A^{-1})_{\sigma(q)}. \tag{3.38}$$

In the same manner, the curvature F_{ij}^c can be shown to transform according to

$$F_{ij}(\tau(q)) = k(q)F_{ij}(\sigma(q)) \qquad F_{ij} = \sum_a F_{ij}^a e_a. \tag{3.39}$$

In fact, from (2.36) along with the local section τ, we observe that

$$\mathrm{d}R\left(\sum_i \beta_i(\tau(q))\,\mathrm{d}q^i\right) - R\left(\sum_i \beta_i(\tau(q))\,\mathrm{d}q^i\right) \wedge R\left(\sum_j \beta_j(\tau(q))\,\mathrm{d}q^j\right)$$
$$= \sum_{i<j} R(F_{ij}(\tau(q)))\,\mathrm{d}q^i \wedge \mathrm{d}q^j \tag{3.40}$$

the left-hand side of which can be verified, on account of (3.33), to be subject to the transformation

$$\mathrm{d}R\left(\sum_i \beta_i(\tau(q))\,\mathrm{d}q^i\right) - R\left(\sum_i \beta_i(\tau(q))\,\mathrm{d}q^i\right) \wedge R\left(\sum_j \beta_j(\tau(q))\,\mathrm{d}q^j\right)$$
$$= \mathrm{Ad}_{k(q)}\left(\mathrm{d}R\left(\sum_i \beta_i(\sigma(q))\,\mathrm{d}q^i\right) - R\left(\sum_i \beta_i(\sigma(q))\,\mathrm{d}q^i\right)\right.$$
$$\left.\wedge R\left(\sum_j \beta_j(\sigma(q))\,\mathrm{d}q^j\right)\right) \tag{3.41}$$

and hence (3.39) follows. From (3.38) and (3.39) it follows that the right-hand side of (3.34) is indeed independent of the choice of local sections.

On account of (3.35), the first and the third terms of the right-hand side of (3.34) are put together to be written as

$$-\sum_k a^{ij}\frac{\partial}{\partial q^j}\left(\frac{1}{2}\sum_{a,b} A_{\sigma(q)}^{ab}\Lambda_a\Lambda_b + V\right) + \sum_k a^{ij}((A_{\sigma(q)}^{-1}\Lambda)\times\Lambda|\beta_j(\sigma(q))). \tag{3.42}$$

Thus equation (3.34) becomes expressed as

$$\frac{\mathrm{d}^2q^i}{\mathrm{d}t^2} + \sum_{j,k}\begin{Bmatrix} i \\ j\,k \end{Bmatrix}\frac{\mathrm{d}q^j}{\mathrm{d}t}\frac{\mathrm{d}q^k}{\mathrm{d}t} = -\sum_j a^{ij}\frac{\partial}{\partial q^j}\left(\frac{1}{2}\sum_{a,b} A_{\sigma(q)}^{ab}\Lambda_a\Lambda_b + V\right)$$
$$+\sum_j a^{ij}((A_{\sigma(q)}^{-1}\Lambda)\times\Lambda|\beta_j(\sigma(q)))$$
$$-\sum_{j,k}\sum_c a^{ik}\Lambda_c F_{jk}^c(\sigma(q))\frac{\mathrm{d}q^j}{\mathrm{d}t}. \tag{3.43}$$

The reduced equations of motion then consist of (3.26) and (3.43). These are reminiscent of Wong's equations (1970), if the terms appearing in (3.42) are dropped from the right-hand side of (3.43). See also Montgomery (1990, 1991) for Wong's equation. In the case of $\Lambda = 0$, equation (3.43) reduces to the usual Newton's equations of motion on the internal space, and equation (3.26) vanishes. We will derive Wong's equations for an optimal control problem of the multi-particle system in the next section.

4. A control problem

We consider the control problem of moving the multi-particle system under the condition of the vanishing total angular momentum. Since a vector field is vibrational if and only if the total angular momentum vanishes (see (2.18)), the equation the system must satisfy can be written as

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \sum_i u^i \partial_i^* x \qquad x \in \dot{X} \tag{4.1}$$

where ∂_i^* are the basis of vibrational vector fields given in (2.27), and u^i are controls, functions of t. Since $\mathrm{d}x = \sum_a \omega^a J_a x + \sum_i \mathrm{d}q^i \partial_i^* x$, equation (4.1) is equivalent to

$$\frac{\mathrm{d}q^i}{\mathrm{d}t} = u^i \qquad \omega_t^a = 0 \tag{4.2}$$

where ω_t^a is given by (3.11). If we are given a curve $q(t)$ in M, equation (4.1) with $u^i = \mathrm{d}q^i/\mathrm{d}t$ determines a horizontal (or vibrational) curve $x(t)$ in $\dot{X}$ such that $\pi(x(t)) = q(t)$. On account of (2.24) and (2.25), the second equation of (4.2) is expressed as

$$\frac{\mathrm{d}g}{\mathrm{d}t} g^{-1} + \sum_i R(\beta_i(g\sigma(q))) \frac{\mathrm{d}q^i}{\mathrm{d}t} = 0 \qquad \beta_i = (\beta_i^a) \tag{4.3}$$

which is put, by the use of (2.15) and (2.28), in the form

$$\frac{\mathrm{d}g}{\mathrm{d}t} = -g \sum_i R(\beta_i(\sigma(q))) \frac{\mathrm{d}q^i}{\mathrm{d}t}. \tag{4.4}$$

Thus (4.2) turns out to be expressed as

$$\frac{\mathrm{d}q^i}{\mathrm{d}t} = u^i \qquad \frac{\mathrm{d}g}{\mathrm{d}t} = -g \sum_i R(\beta_i(\sigma(q))) u^i. \tag{4.5}$$

To define an optimal control problem associated with (4.5), we have to provide a performance index. Since we are to consider horizontal paths subject to (4.5), we assume that $\dot{X}$ is endowed only with a 'horizontal metric', which comes from (2.33) to be defined as

$$\mathrm{d}s_0^2 = \sum_{i,j} a_{ij} \, \mathrm{d}q^i \, \mathrm{d}q^j. \tag{4.6}$$

To be strict, to use the word 'metric' for $\mathrm{d}s_0^2$ is not adequate, since $\mathrm{d}s_0^2$ is degenerate as a quadratic form. However, we call $\mathrm{d}s_0^2$ the horizontal metric for convenience. Now a performance index is defined to be

$$\frac{1}{2} \int_0^T \langle u(t), u(t) \rangle_{x(t)} \, \mathrm{d}t \tag{4.7}$$

where $\langle\ ,\ \rangle$ and $u(t) = \sum_i u^i(t) \partial_i^*$ denote the horizontal metric (4.6) and a horizontal tangent vector to $x(t)$. Our optimal control problem is now set up as a problem of determining controls $u^i(t)$ in such a way that the performance index (4.7) is minimized among all the controls which steer the state $x(t)$ from an initial state x_0 to a final state x_1 in time T. It should be noted here that x_0 and x_1 are chosen so that they may be joined by a horizontal curve, in order that our problem is well set up. However, in our case, any pair of points of $\dot{X}$ can be joined by a horizontal curve, as was shown by Guichardet (1984).

In order to apply the Maximum Principle, we consider the control problem on the cotangent bundle $T^*\dot{X}$. Let θ be the canonical one-form on $T^*\dot{X}$, which is defined, as usual, to be

$$\theta = \sum_{\alpha=1}^{N} (p_\alpha | \mathrm{d}x_\alpha) \qquad (x, p) \in T^*\dot{X} \qquad x = (x_\alpha),\ \ p = (p_\alpha). \tag{4.8}$$

According to the local triviality (2.6), the θ is expressed as

$$\theta = \sum_a \gamma_a \Xi^a + \sum_i p_i \, \mathrm{d}q^i \tag{4.9}$$

where we have set

$$\gamma = \sum_{\alpha=1}^{N} \sigma_\alpha(q) \times g^{-1} p_\alpha = \sum_a \gamma_a e_a \tag{4.10}$$

$$p_i = \sum_{\alpha=1}^{N} \left(g^{-1} p_\alpha \middle| \frac{\partial \sigma_\alpha}{\partial q^i} \right) \tag{4.11}$$

$$g^{-1}\,\mathrm{d}g = \sum_{a=1}^{3} \Xi^a R(e_a). \tag{4.12}$$

The momentum variables associated with ∂_i^* are then defined and expressed as

$$P_i := \theta(\partial_i^*) = p_i - \sum_{c=1}^{3} \beta_i^c(\sigma(q))\gamma_c. \tag{4.13}$$

In terms of P_i, the canonical one-form θ is put in the form

$$\theta = \sum_a L'_a \omega^a + \sum_i P_i \, \mathrm{d}q^i \tag{4.14}$$

where

$$L'_a = \sum_b g_{ab}\gamma_b \tag{4.15}$$

the components of the total 'angular momentum'

$$L' = \sum_{\alpha=1}^{N} x_\alpha \times p_\alpha = g \sum_{\alpha=1}^{N} \sigma_\alpha(q) \times g^{-1} p_\alpha = g\gamma. \tag{4.16}$$

The total angular momentum L' is not related to the mechanical total angular momentum

$$L = \sum_{\alpha=1}^{N} m_\alpha x_\alpha \times \dot{x}_\alpha. \tag{4.17}$$

If

$$m_\alpha \dot{x}_\alpha = p_\alpha \qquad \alpha = 1, \ldots, N \tag{4.18}$$

then we would obtain $L = L'$. We should note here that the relation (4.18) provides the isomorphism of $T\dot{X}$ to $T^*\dot{X}$ through the metric $\mathrm{d}s^2 = \sum_\alpha m_\alpha (\mathrm{d}x_\alpha | \mathrm{d}x_\alpha)$. However, $\mathrm{d}s^2$ is not required to be endowed with $\dot{X}$ in our control problem. Only the horizontal metric (4.6) is needed to set up our control problem, so that (4.18) has no reason to hold, and therefore L' is not related to L.

Now that we have set up the phase space $(T^*\dot{X}, \mathrm{d}\theta)$, the Maximum Principle tells us that an optimal control $u = (u^i)$ is to be determined so that the Hamiltonian

$$\mathcal{H} = \sum_i P_i u^i - \frac{1}{2}\langle u, u\rangle \tag{4.19}$$

may be maximized, where u is looked upon as a vibrational vector, $u = \sum_i u^i \partial_i^*$. In the case of normal extremals, we find that an optimal control is given by $u^i = \sum_j a^{ij} P_j$. The Hamiltonian (4.19) then takes the form

$$H = \frac{1}{2}\sum_{i,j} a^{ij} P_i P_j. \tag{4.20}$$

Thus we obtain the Hamiltonian system $(T^*\dot{X}, \mathrm{d}\theta, H)$ arising from the optimal control problem for the multi-particle system, which is manifestly $SO(3)$-invariant. The equations of motion are obtained from the Hamiltonian vector field X_H determined by $\iota(X_H)\,\mathrm{d}\theta = -\mathrm{d}H$. By using (4.9) and (4.20), we obtain the equations

$$\begin{aligned} &\frac{\mathrm{d}q^i}{\mathrm{d}t} = \frac{\partial H}{\partial p_i} \qquad \frac{\mathrm{d}p_i}{\mathrm{d}t} = -\frac{\partial H}{\partial q^i} \\ &\Xi_t^a = \frac{\partial H}{\partial \gamma_a} \qquad \frac{\mathrm{d}\gamma_a}{\mathrm{d}t} = \sum_{b,c} \varepsilon_{cba}\gamma^c \frac{\partial H}{\partial \gamma_b} \end{aligned} \tag{4.21}$$

where $\Xi_t^a = \Xi^a(\mathrm{d}/\mathrm{d}t)$. Written out, these equations become

$$\begin{aligned} \frac{\mathrm{d}q^i}{\mathrm{d}t} &= \sum_j a^{ij} P_j \\ \frac{\mathrm{d}p_i}{\mathrm{d}t} &= \sum_{k,j}\sum_c a^{kj} \frac{\partial \beta_k^c(\sigma(q))}{\partial q^i} P_j \gamma_c - \frac{1}{2}\sum_{k,j} \frac{\partial a^{kj}}{\partial q^i} P_k P_j \\ \Xi_t^a &= -\sum_{i,j} a^{ij} P_i \beta_j^a(\sigma(q)) \\ \frac{\mathrm{d}\gamma_a}{\mathrm{d}t} &= -\sum_{b,c}\sum_i \varepsilon_{abc}\gamma_c a^{ij} P_i \beta_j^b(\sigma(q)) \end{aligned} \tag{4.22}$$

where P_i are the momentum variables defined in (4.13). Three of these equations are put together to be rewritten as

$$\begin{aligned} &\frac{\mathrm{d}^2 q^i}{\mathrm{d}t^2} + \sum_{j,k} \left\{ {i \atop j\,k} \right\} \frac{\mathrm{d}q^j}{\mathrm{d}t}\frac{\mathrm{d}q^k}{\mathrm{d}t} = \sum_{j,k}\sum_c F_{jk}^c(\sigma(q)) a^{ij} \frac{\mathrm{d}q^k}{\mathrm{d}t}\gamma_c \\ &\frac{\mathrm{d}\gamma}{\mathrm{d}t} = -\gamma \times \sum_i \beta_i(\sigma(q)) \frac{\mathrm{d}q^i}{\mathrm{d}t}. \end{aligned} \tag{4.23}$$

Since the γ is a locally-defined variable, we have to verify that these equations are independent of the choice of local sections. If we take the local section τ given in (2.7), γ should be replaced by

$$\overline{\gamma} = h^{-1}L' \tag{4.24}$$

which corresponds to (4.16). Then equation (2.8) gives rise to

$$\overline{\gamma} = k\gamma. \tag{4.25}$$

Equations (3.39) and (4.25) show that the right-hand side of the first equation of (4.23) is independent of the choice of local sections. As for the second equation of (4.23), one can verify that

$$\frac{\mathrm{d}\overline{\gamma}}{\mathrm{d}t} = -\overline{\gamma} \times \sum_i \beta_i(\tau(q)) \frac{\mathrm{d}q^i}{\mathrm{d}t} \tag{4.26}$$

on account of (3.33) and (4.25). Thus we observe that the equations of motion (4.23) are independent of the choice of local sections.

Equations (4.23) are the Wong equations describing the motion of a classical particle in the Yang–Mill field F^c_{jk}, to which Montgomery (1990, 1991) has already referred, without explicit calculation. The remaining equation containing Ξ^a_t in (4.22) is concerned with the rotational variables, and turns out to be expressed, on account of (4.12), as

$$\frac{\mathrm{d}g}{\mathrm{d}t} = -g \sum_{j,k} R(\beta_j(\sigma(q))a^{jk} P_k \tag{4.27}$$

which is equation (4.5) with $u^i = \sum_k a^{ik} P_k$ and can be integrated after equation (4.23) is solved. Equation (4.27) can be put in the form

$$g^{-1}\frac{\mathrm{d}g}{\mathrm{d}t} + \sum_j R(\beta_j(\sigma(q))\frac{\mathrm{d}q^j}{\mathrm{d}t} = 0 \tag{4.28}$$

which is equivalent, under (2.6), to

$$L = \sum_{\alpha=1}^{N} m_\alpha x_\alpha \times \frac{\mathrm{d}x_\alpha}{\mathrm{d}t} = 0. \tag{4.29}$$

Further, the angular momentum L' is shown to be conserved on account of (4.23). However, it depends on the initial condition whether L' vanishes or not. This does not contradict (4.29), since L and L' need not be equal.

Equations (4.23) are considered as reduced equations by the $SO(3)$-symmetry from the Hamilton equations on $T^*\dot{X}$, and looked upon as defined on $T^*\dot{X}/SO(3) \cong T\dot{X}/SO(3) \cong T(M) \oplus \mathrm{Ad}(\dot{X})$ with $\mathrm{Ad}(\dot{X}) := \dot{X} \times_{SO(3)} \mathcal{G}$ and $\mathcal{G} = so(3)$. It is of great interest to compare equation (4.23) with the reduced equations of motion, (3.26) and (3.43), for the mechanical system. If we could set $A = 0$ and replace γ for Λ in equations (3.26) and (3.43), we would obtain the Wong equations (4.23). This comparison of equations would allow for the interpretation that the choice of velocities as control variables implies conversely that the control system is assumed to have vanishingly small inertia. However, we note that γ and Λ are comparable, but not equal.

5. Remarks

We have applied the Lagrangian equations (3.6), of motion to the Lagrangian system for the multi-particle system. We remark, in conclusion, that equation (3.6) is also applicable to Lagrangian systems on the tangent bundle of any principal fibre bundle. Let X be a principal fibre bundle with structure group G acting on X to the left. We take the local triviality of this bundle as $\pi^{-1}(U) \cong U \times G$ with local coordinates (q, g), which is similar to (2.6). We assume that a connection ω is defined on X, which satisfies a similar equation to (2.19) and is subject to the same transformation as (2.20) with $g \in G$. Then the components

ω^a, $a = 1, \ldots, r = \dim \mathcal{G}$, take the same expression as (2.24), $\mathcal{G}$ being the Lie algebra of G. Let C^c_{ab} be the structure constants of $\mathcal{G}$ along with the structure equation

$$[E_a, E_b] = \sum_{c=1}^{r} C^c_{ab} E_c \tag{5.1}$$

for a basis $E_a, a = 1, \ldots, r$. Then, in our present case, equation (2.36) is put in the form

$$\mathrm{d}\omega^c = \sum_{a<b} C^c_{ab} \omega^a \wedge \omega^b + \sum_{i<j} F^c_{ij}\, \mathrm{d}q^i \wedge \mathrm{d}q^j \tag{5.2}$$

where F^c_{ij} are the components of the curvature defined to be

$$F^c_{ij} = \frac{\partial \beta^c_j}{\partial q^i} - \frac{\partial \beta^c_i}{\partial q^j} - \sum_{a,b=1}^{r} C^c_{ab} \beta^a_i \beta^b_j. \tag{5.3}$$

We notice here that, while in Koiller (1992) the functions β^a_j are treated as independent of $g \in G$, in our case they depend on g.

On the above setting, we take a basis of the space of one-forms on $\pi^{-1}(U)$ as

$$\theta^a = \omega^a \qquad \theta^{3+i} = \mathrm{d}q^i \qquad a = 1, \ldots, r = \dim \mathcal{G},\ i = 1, \ldots, n = \dim M \tag{5.4}$$

and the dual basis as

$$X_a = J_a \qquad X_{r+i} = \partial^*_i \qquad a = 1, \ldots, r = \dim \mathcal{G},\ i = 1, \ldots, n = \dim M \tag{5.5}$$

where

$$J_a = \frac{\mathrm{d}}{\mathrm{d}t} \exp(tE_a)x|_{t=0} \qquad \partial^*_j = \frac{\partial}{\partial q^j} - \sum_{a=1}^{r} \beta^a_j J_a \tag{5.6}$$

are the infinitesimal transformation of $\exp(tE_a)$ and the horizontal lift of $\partial/\partial q^j$, respectively. Then, like (3.9), one has

$$\gamma^a_{bc} = -C^a_{bc} \qquad \gamma^a_{r+i,r+j} = -F^a_{ij} \tag{5.7}$$

with the other coefficients $\gamma^\lambda_{\mu\nu}$ vanishing. Hence, the Lagrangian equations (3.13) and (3.14) for a certain Lagrangian L^* take the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L^*}{\partial \omega^a_t}\right) - J_a L^* - \sum_{b,c} C^b_{ac} \frac{\partial L^*}{\partial \omega^b_t} \omega^c_t = 0 \tag{5.8}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L^*}{\partial \dot{q}^i}\right) - \partial^*_i L^* - \sum_a \sum_j F^a_{ij} \frac{\partial L^*}{\partial \omega^a_t} \dot{q}^j = 0 \tag{5.9}$$

respectively, where ω^a_t are defined in the same manner as the previous one. Though our system is not a non-holonomic Lagrangian system, our Lagrangian formulation has a resemblance to that of a non-holonomic system. See de León and de Diego (1996) and references therein for a treatment of non-holonomic Lagrangian systems.

The Hamiltonian formalism runs as follows: let T^*X be the cotangent bundle of X endowed with the canonical one-form θ. We determine the momentum variables π_a and P_i in a manner such that θ is expressed as

$$\theta = \sum_a \pi_a \omega^a + \sum_i P_i\, \mathrm{d}q^i \tag{5.10}$$

where ω^a are the components of the connection form. The canonical symplectic form $\mathrm{d}\theta$ then takes the form

$$\mathrm{d}\theta = \sum_a \mathrm{d}\pi_a \wedge \omega^a + \sum_i \mathrm{d}P_i \wedge \mathrm{d}q^i + \sum_a \sum_{b<c} \pi_a C^a_{bc} \omega^b \wedge \omega^c + \sum_a \sum_{i<j} \pi_a F^a_{ij} \,\mathrm{d}q^i \wedge \mathrm{d}q^j \tag{5.11}$$

where use has been made of (5.2). When given a Hamiltonian function H^*, the associated Hamiltonian vector field X_{H^*} is defined, as usual, through $\iota(X_{H^*})\,\mathrm{d}\theta = -\mathrm{d}H^*$, and turns out to take the form

$$\begin{aligned} X_{H^*} = {} & \sum_a \frac{\partial H^*}{\partial \pi_a} J_a - \sum_a \left(\sum_{b,c} C^c_{ba} \pi_c \frac{\partial H^*}{\partial \pi_b} + J_a(H^*) \right) \frac{\partial}{\partial \pi_a} \\ & + \sum_i \frac{\partial H^*}{\partial P_i} \partial^*_j - \sum_j \left(\sum_a \sum_i \pi_a F^a_{ij} \frac{\partial H^*}{\partial P_i} + \partial^*_j H^* \right) \frac{\partial}{\partial P_j}. \end{aligned} \tag{5.12}$$

The Hamiltonian equations of motion are then expressed as

$$\begin{aligned} \omega^a_t &= \frac{\partial H^*}{\partial \pi^a} \\ \frac{\mathrm{d}\pi_a}{\mathrm{d}t} &= -\sum_{b,c} C^c_{ab} \pi_c \frac{\partial H^*}{\partial \pi_b} - J_a(H^*) \\ \frac{\mathrm{d}q^i}{\mathrm{d}t} &= \frac{\partial H^*}{\partial P_i} \\ \frac{\mathrm{d}P_j}{\mathrm{d}t} &= -\sum_a \sum_i \pi_a F^a_{ij} \frac{\partial H^*}{\partial P_i} - \partial^*_j H^*. \end{aligned} \tag{5.13}$$

To look into the right-hand side of the second equation of (5.13), we note that the lift, $\widetilde{J}_a$, of the infinitesimal transformation J_a is defined to be an infinitesimal transformation satisfying

$$\mathcal{L}_{\tilde{J}_a} \theta = 0 \qquad \mathrm{pr}_* \widetilde{J}_a = J_a \tag{5.14}$$

where pr is the projection $T^*X \to X$, and pr_* is its differential. From the formula

$$\mathcal{L}_{J_a} \omega^b = \sum_c C^b_{ac} \omega^c \tag{5.15}$$

which is a consequence of (5.1) and (2.20) with $g \in G$, it turns out that

$$\widetilde{J}_a = J_a + \sum_{b,c} C^c_{ba} \pi_c \frac{\partial}{\partial \pi_b}. \tag{5.16}$$

Therefore, the second equation of (5.13) is expressed as $\mathrm{d}\pi_a/\mathrm{d}t = -\widetilde{J}_a(H^*)$, so that if the Hamiltonian H^* is invariant under the lifted action of G on T^*X, $\widetilde{J}_a(H^*) = 0$, then π_a are conserved. This fact is a generalization of the conservation of the total angular momentum. In fact, for $G = SO(3)$, one has $\omega^a = \Theta^a + \sum_i \beta^a_i \,\mathrm{d}q^i$ with $\Theta^a = \sum_b g_{ab} \Xi^b$, so a comparison of (4.14) with (5.10) shows that $\pi_a = L'_a$ together with $\pi_a = \sum_b g_{ab} \gamma_b$. In particular, for $H^* = 1/2 \sum_{i,j} a^{ij} P_i P_j$, we obtain from (5.13) the equations of motion equivalent to (4.22).

It is to be noted that if starting with the Lagrangian $L^* = 1/2 \sum_{i,j} a_{ij} \dot{q}^i \dot{q}^j$ (to be compared with (3.12)), we can define $P_i = \partial L^*/\partial \dot{q}^i$, but cannot use $\partial L^*/\partial \omega^a_t$ to define conjugate momenta, since $\partial L^*/\partial \omega^a_t = 0$. In this case, however, the Maximum Principle can provide the Hamiltonian H^* in the form $H^* = 1/2 \sum_{i,j} a^{ij} P_i P_j$ (see (4.20)).

References

Abraham R and Marsden J E 1978 *Foundations of Mechanics* (Reading MA: Benjamin)
Cushman R H and Bates L M 1997 *Global Aspects of Classical Integrable Systems* (Basel: Birkhäuser)
Guichardet A 1984 *Ann. Inst. Henri Poincaré* **40** 329–42
Iwai T 1987a *Ann. Inst. Henri Poincaré* **47** 199–219
——1987b *J. Math. Phys.* **28** 1315–26
Koiller J 1992 *Arch. Ration. Mech. Anal.* **118** 113–48
de León M and de Diego D M 1996 *J. Math. Phys.* **37** 3389–414
Littlejohn R G and Reinsch M 1997 *Rev. Mod. Phys.* **69** 213–75
Matsushima Y 1972 *Differentiable Manifolds* (New York: Dekker)
Montgomery R 1990 *Commun. Math. Phys.* **128** 565–92
——1991 *The Geometry of Hamiltonian Systems* ed T Ratiu (New York: Springer) pp 403–38
Neimark Ju and Fufaev N A 1972 Dynamics of nonholonomic systems *Am. Math. Soc. Transl.* **33**
Whittaker E T 1937 *Analytical Dynamics of Particles and Rigid Bodies* (Cambridge: Cambridge University Press)
Wong S K 1970 *Nuovo Cimento* **65** 689–94